Quantum Fields in Expanding Universe
Mukanov and Winitzki Chap 6 Notes

Robert D. Klauber October 22, 2016 www.quantumfieldtheory.info

Refs:
Introduction to Quantum Effects in Gravity, Mukhanov, V., and Winitzki, S. (Cambridge, 2007)
Student Friendly Quantum Field Theory, Klauber, R.D., (Sandtrove 2015, 2nd ed, 3rd printing)

NOTE: Section numbers, headings, and equation numbers of form (6.X) are with reference to Mukhanov & Winitzki (M&W).

TYPOS:

Pg. 66, (6.14), RHS: v's inside parentheses need subscript k.
Pg. 67, top line: same correction as above

6.1 Classical scalar field in expanding background

See Klauber "Conformal and Scale Invariant Transformations Simplified" (available at www.quantumfieldtheory.info) for simplified, background understanding of coordinate vs physical values, and additionally, for the reason why η used in M&W is called "conformal time".

$\mathbf{x},\eta$ are coordinate values. ($\mathbf{x}$ are fixed position values attached, for example, to the centers of galaxies to form a co-moving 3D coordinate system[1]). $d\mathbf{x},d\eta$ are coordinate differences between events. ($d\mathbf{x}$ is the difference in position values, for example, between two galaxy centers infinitesimally [hypothetically] close together). $a(t)dx^i$ and $dt = a(t)\eta$ are physical values ($a(t)dx^i$ is what would be measured with meter sticks between galaxy centers in the ith direction[2]). See Fig. 1.

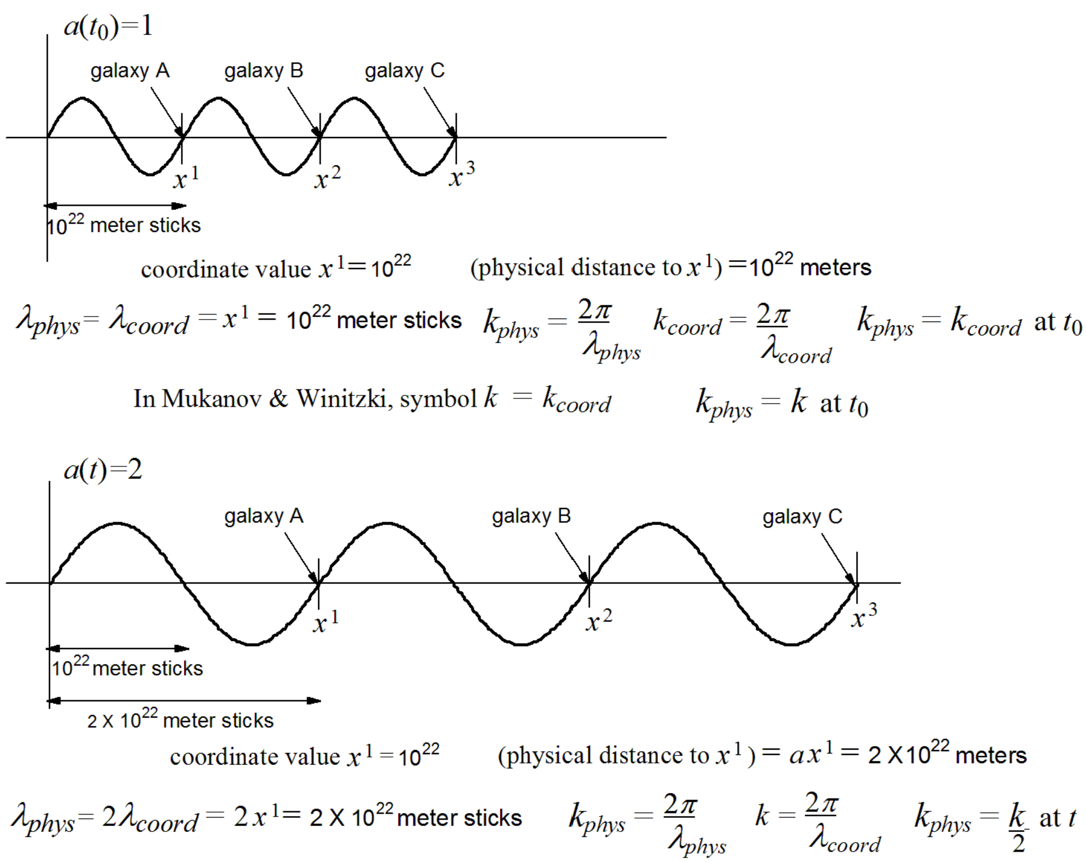

Figure 1. Coordinate Values vs Physical Values in an Expanding Universe

---

[1] Or, in the example in Klauber "Conformal and Scale Invariant Transformations Simplified" article, to values painted on coordinate lines that are fixed to the stretchable rubber sheet as the sheet stretches. These are often called "co-moving coordinates" (because they move with the material points).

[2] Or in the rubber sheet example, to the distance measured with meter sticks that do not stretch with the rubber sheet.

Note that $a(t)$ has units of meters per coordinate grid value. (Physical length between points) = $a(t)$ X (coordinate grid label difference value between those points).

In (6.3) (of M&W) S is the physical (what would be measured with our instruments) action. d^4x is coordinate 4-volume, and $\sqrt{-g}d^4x$ is physical 4-volume. $\sqrt{-g} = a^4$.

$$\tfrac{1}{2}\left(g^{\alpha\beta}\phi_{,\alpha}\phi^{,\alpha} - m^2\phi^2 = \underset{\sim}{\mathcal{L}}\right) \qquad \text{of (6.3) is the physical Lagrangian density for field } \phi \tag{1}$$

with units of physical energy per physical 3-volume (in non-natural units, Joules/m^3 or Ft-lbs/ft^3, for examples). The action S has dimensions of energy-time (in non-natural MKS units Joule-sec)[3].

In (6.4), $d^3\mathbf{x}d\eta$ is coordinate 4-volume d^4x, the magnitude of which is independent of $a(t)$.

$$\tfrac{1}{2}a^2\left(\phi'^2 - \left(\nabla^2\phi\right)^2 - m^2a^2\phi^2\right) = \mathcal{L} \quad \text{of (6.4) is the coordinate Lagrangian density for field } \phi \tag{2}$$

having derivatives with respect to coordinates $\mathbf{x}$ and η, not physical distance a(t)x or physical time t. Units of the coordinate Lagrangian are now hybrid as they represent an energy in terms of non-physical (coordinate) time per unit non-physical (coordinate) 3-volume. However, importantly, when (2) is integrated with respect to $d^3\mathbf{x}d\eta$, the result is still the action S with dimensions of energy-time (in non-natural MKS units, Joule-sec).

In (6.5), the field χ is the physical (amplitude for a classical wave like the wave of Fig. 1) field; ϕ is the coordinate field (the physical field value at coordinate time $\eta = \eta_0$, which is when we typically take physical time $t = 0$ [t_0 in Fig. 1], which is when $a = 1$). Classically, ϕ at a given fixed material location (like the center of a galaxy) does not change in magnitude as the universe expands, like the coordinate value $\mathbf{x}$ at such fixed material location does not change in value during the expansion. (The center of the galaxy keeps the same coordinate location values $\mathbf{x}$, while the physical location changes. That galaxy center also keeps the same ϕ value, even as its χ value grows.) So as the universe grows in size proportional in any direction to $a(t)$, the classical (physical) field χ amplitude grows as well (as in Fig. 1), and both are proportional to $a(t)$ (which can also be expressed as $a(\eta)$).

In quantizing a classical field, one realizes that the magnitude of the classical (bosonic) field is the sum of many quanta. More quanta means a higher amplitude resultant classical field. The amplitude of the quanta are fixed because the waves are normalized (to make our probability and number operator calculations come out correctly.) So for a given $\mathbf{k}$ eigenstate, more $\mathbf{k}$ quantum states superimposed yield the higher the amplitude of the resultant classical wave.

As one might guess, therefore, and as will be seen, if the classical χ increases with $a(t)$, then the quantized χ, which is comprised of creation and destruction operators, will create more $\mathbf{k}$ states, for each value of $\mathbf{k}$. This should (and it does, as will be seen) increase the energy of the universe.

This jibes with classical mechanics, because energy is only conserved if the Hamiltonian is not an explicit function of time. If $H = H(t)$, then energy is not conserved (it changes in time.) In (6.4) we see the mass term has a factor of $(a(t))^2$ [or equivalently, $(a(\eta))^2$] in the Lagrangian density. Since we get the Hamiltonian density from the Lagrangian density via the Legendre transformation, and the Hamiltonian from the Hamiltonian density, the Hamiltonian will be an explicit function of time. Thus, energy is not conserved.[4]

In (6.6), $d^3\mathbf{x}\, d\eta$ is coordinate 4-volume. The quantity

$$\tfrac{1}{2}\left(\chi'^2 - (\nabla\chi)^2 - \left(m^2a^2 - \frac{a''}{a}\right)\chi^2\right) \quad \text{of (6.6) is the coordinate Lagrangian density for field } \chi \tag{3}$$

analogous to that of (6.4), or (2) herein. (3) has the same (unusual) units as (2), and the same numerical value at each event in 4D spacetime. It is just expressed in terms of the field χ instead of the field ϕ. And again, S in (6.6) has the usual, familiar, physical dimensions of energy-time (joule-sec in MKS units).

---

[3] In natural units S is dimensionless.

[4] (6.3) to (6.6) are for free fields, but in truth, the fields in the universe are subject to the gravitational potential, which also changes at a material point (point in the co-moving coordinate system of $\mathbf{x}$) as the universe expands. One might wonder if this would cancel with the creation of energy due to expansion. However, it seems the gravitational potential grows less negative (increases) as the universe expands, so it would seem to add to the increase in energy anticipated here.

Using hybrid Lagrangians and Hamiltonians to get (6.7)

Note that the entire theory of variational mechanics follows from any Lagrangian density in any form with respect to any coordinates. That particular Lagrangian density integrated out over spacetime yields the action S to be used in our theory. This can be done in many different ways from classical fields to quantum fields to economic models. It is a very general procedure.

So, in the present case, we can still use the Euler-Lagrange equations to get our field equation. This is what is done to get (6.7).

Caution on energy units when using coordinate spacetime values

And we can still use the Legendre transformation to get energy and energy density. However, we must be aware that the resultant values we get for energy and energy density will have hybrid units.

Note that the spatial effect of the energy density per unit coordinate volume (instead of per unit physical volume) will integrate will disappear when we integrate over the volume (integration over coordinate distances) to get total energy. However, the effect of using coordinate time η in our time derivatives of the field (ϕ or χ here) will still be present in our total energy values. We would need to convert the resultant energy value found based on coordinate time to its equivalent in physical time in order to compare our results with what our physical measuring instruments would measure.

Note on wave number **k**

Note that **k**, as used in M&W, is the coordinate wave number. A given wave, as it stretches with the expansion of the universe will keep the same coordinate wave number **k** as its label, but its physical wave number, which we would measure with instruments would vary inversely with $a(t)$. A longer wavelength wave, has a lower (physical) wave number. Thus,

$$\begin{aligned} &\lambda_{phys} = a(t)\lambda \qquad \lambda \text{ is coordinate wavelength } \left(= \text{physical wavelength at } a(t=t_0)=1\right) \\ &k = \frac{2\pi}{\lambda} \\ &\mathbf{k}_{phys} = \frac{\mathbf{k}}{a(t)} \qquad \text{coord value } \mathbf{k} \text{ unchanged for same wave as its wavelength stretches; } \mathbf{k}_{phys} \text{ lessens} \end{aligned} \quad . \tag{4}$$

In M&W, the theory is developed with respect to the coordinate values $\mathbf{x},\eta$, not the physical values of actual distance and time. This is done to simplify the analysis. We can always convert our answers in coordinate values back to physical values by multiplying or dividing by $a(\eta)$.

Note further that since

$$\text{physical distance between origin and a coord pt} = \int_0^{\mathbf{x}} a(t)d\mathbf{x}' = a(t)\mathbf{x} = \mathbf{x}_{phys}\,, \tag{5}$$

$$\mathbf{k}\cdot\mathbf{x} = \frac{\mathbf{k}}{a(t)}\cdot a(t)\mathbf{x} = \mathbf{k}_{phys}\cdot\mathbf{x}_{phys}\,. \tag{6}$$

Thus,

$$e^{i\mathbf{k}\cdot\mathbf{x}} = e^{i\mathbf{k}_{phys}\cdot\mathbf{x}_{phys}} = \text{ same value expressed via physical or coordinate values}\,, \tag{7}$$

which may help when we consider the general wavelike form of any field.

6.1.1 Mode expansion

The Spatial Fourier Mode Expansion

For (6.9), in elementary QFT, when our Lagrangian (and Hamiltonian) was not a function of time, we had

$$\chi_{\mathbf{k}}(\eta) = \chi_{\mathbf{k}}(t) \propto e^{\pm i\omega_{\mathbf{k}}t} \qquad \text{for } a(t)=1 \text{ constant } \left(\text{no time dependence in Hamiltonian}\right). \tag{8}$$

However, since we do not have constant a (so Hamiltonian is a function of time), we need to consider what a more general form for $\chi_{\mathbf{k}}$ would look like. This is done on pgs 72-73, but for the next few pages in M&W, the general form (6.13) for it is represented by the symbol $\chi_{\mathbf{k}}(\eta)$.

If H were not a function of time, then (6.13) would take a simple form familiar from QFT.

$$\chi_{\mathbf{k}}(\eta) = \chi_{\mathbf{k}}(t) = \frac{1}{\sqrt{2}}\left(a_{\mathbf{k}}^{-} v_k^{*} + a_{-\mathbf{k}}^{+} v_k \right) = \frac{1}{\sqrt{2}}\left(a_{\mathbf{k}}^{-} e^{-i\omega_{\mathbf{k}}t} + a_{-\mathbf{k}}^{+} e^{i\omega_{\mathbf{k}}t} \right)$$
$$v_k = \frac{1}{\sqrt{\omega_{\mathbf{k}}}} e^{i\omega_{\mathbf{k}}t} \qquad a(t) = 1 \text{ constant } \; (\text{no time dependence in } H) \tag{9}$$

Normalization via the Wronskian

The Wronskian normalization of v_k, for each **k**, is

$$v_k' v_k^{*} - v_k v_k^{*\prime} = 2i \qquad \text{M\&W (6.14) with RHS = normalization } 2i \tag{10}$$

Upon seeing the normalization of taking (6.14) = 2*i*, one might immediately ask "where does this come from?" or "why is this being used?".

The authors don't reference it here, but earlier on page 48, where (4.27) is the Wronskian, they do give a reason. Only with (6.14) = 2*i* are the commutation relations ((4.24) and (6.23)) satisfied.

Another perspective on this follows from Klauber, Sect 3.1.4, pgs. 44-47. We need to normalize ϕ there so the RQM probability density, which is

$$\rho = j^0 = i\left(\frac{\partial \phi}{\partial t}\phi^{\dagger} - \frac{\partial \phi^{\dagger}}{\partial t}\phi\right) \qquad (3-20) \text{ in Klauber }, \tag{11}$$

integrates out over all space to one.

(11) looks like the Wronskian of M&W (6.14). In Klauber ϕ was normalized using the integral of (11) equal to one for the discrete solutions case. In M&W we are dealing with continuous solutions. If we followed the integration as done in Klauber for continuous solutions we would find (11) integrates out over all space to unity for a given value of **k**.

For QFT, rather than RQM, we would find the integral of (11) acting on a single particle state would give us an eigenvalue of 1. For an *n* particle state, it would give us *n*. Thus, for our number operator to come out correctly, we need a normalization using (11), i.e., a normalization using the Wronskian.

This is why we need a normalization of (6.14) in M&W to equal 2*i*. If we do that, our number operators come out correctly to equal $a_{\mathbf{k}}^{+} a_{\mathbf{k}}^{-}$, and thus the whole rest of the theory (built up from the number operator relation) will work out correctly.

6.2 Quantization

Note that on page 68, in the remark section about a third of the way down the page, $b_{\mathbf{k}}^{+}$ is not the same as the same symbol $b_{\mathbf{k}}^{+}$ used in (6.26) and the rest of the chapter. In the former case it represents the creation operator for antiparticles of a complex (charged) field. In (6.26) it represents the creation operator for a new real (not charged) field obtained via the Bogolyubov transformation.

6.4 Hilbert Space: "*a* and *b* particles"

Note on (6.32)

The last line of (6.32), pg. 70, results from the commutation relations for the *b* operators.

$\delta^3(0)$ accounting for ∞ spatial volume

On pg. 70, (6.32), the $\delta^3(0)$ factor is said to "account for an infinite spatial volume". M&W discuss this a bit on pg. 50, though they don't reference that earlier stuff in this section. Klauber discusses this on pg. 454 at the top third of the page for a 4D volume, which is analogous to the 3D case. In short,

$$\delta^3(0) = \lim_{V\to\infty} \frac{V}{(2\pi)^3}, \tag{12}$$

which is derivable from

$$\delta^3(\mathbf{p}' - \mathbf{p}) = \frac{1}{(2\pi)^3}\int e^{i(\mathbf{p}'-\mathbf{p})\cdot\mathbf{x}} d^3\mathbf{x} \xrightarrow[\mathbf{p}'=\mathbf{p}]{\text{for}} \delta^3(0) = \frac{1}{(2\pi)^3}\int e^0 d^3\mathbf{x} = \frac{V}{(2\pi)^3} \; (\text{where } V = \infty). \tag{13}$$

General comments

For non time dependent H (as in elementary QFT), "a" or "b" particles (b particle is not an antiparticle here as in usual QFT, but a Bogolyubov transformed real field as in M&W Sect. 6.3) have different vacuums, but whichever you choose, a or b, to represent the universe, you get exactly the same relations. So a is equivalent to b. Using either one to represent physical particles yields the same theory.

For time dependent H, the vacuum without a particles at one time (not counting half quanta) has a particles at a later time. (Also, the half quanta change, as well.)

For the later time, we could find a "b vacuum" particle interpretation with no real b particles (but with half quanta of b particles as in usual QFT for a particles). We need the Bogolyubov transformation to relate the a particle interpretation to the b particle interpretation. Choosing a particles results in a different theory from choosing b particles, unlike elementary QFT.

Squeezed states

The b vacuum expressed as a superposition of excited a particle states is called a "squeezed state". Similarly, the a vacuum can be a squeezed state of excited b particle states. M&W derive this in the solution to Exercise 6.5 (back of book).

Relation (6.33) can be written out in a little more clear form as

$$
\begin{aligned}
\left|{}_{(b)}0\right\rangle &= \prod_{\mathbf{k}} \frac{1}{\left|\alpha_k\right|^{1/2}} e^{\frac{\beta_k}{2\alpha_k}\hat{a}^+_{\mathbf{k}}\hat{a}^+_{-\mathbf{k}}} \left|{}_{(a)}0\right\rangle = \prod_{\mathbf{k}} \frac{1}{\left|\alpha_k\right|^{1/2}} \left(\sum_{n=0}^{\infty} \left(\frac{\beta_k}{2\alpha_k}\right)^n \left(\hat{a}^+_{\mathbf{k}}\hat{a}^+_{-\mathbf{k}}\right)^n \right) \left|{}_{(a)}0\right\rangle \\
&= \prod_{\mathbf{k}} \frac{1}{\left|\alpha_k\right|^{1/2}} \left(\sum_{n=0}^{\infty} \left(\frac{\beta_k}{2\alpha_k}\right)^n \left|{}_{(a)}n_{\mathbf{k}}, n_{-\mathbf{k}}\right\rangle \right) \\
&= \left(\frac{1}{\left|\alpha_{k_1}\right|^{1/2}} \left(\left|{}_{(a)}0\right\rangle + \frac{\beta_{k_1}}{2\alpha_{k_1}} \left|{}_{(a)}1_{\mathbf{k}_1}, 1_{-\mathbf{k}_1}\right\rangle + \left(\frac{\beta_{k_1}}{2\alpha_{k_1}}\right)^2 \left|{}_{(a)}2_{\mathbf{k}_1}, 2_{-\mathbf{k}_1}\right\rangle \right) + \ldots\ldots \right) \times \\
&\quad \left(\frac{1}{\left|\alpha_{k_2}\right|^{1/2}} \left(\left|{}_{(a)}0\right\rangle + \frac{\beta_{k_2}}{2\alpha_{k_2}} \left|{}_{(a)}1_{\mathbf{k}_2}, 1_{-\mathbf{k}_2}\right\rangle + \left(\frac{\beta_{k_2}}{2\alpha_{k_2}}\right)^2 \left|{}_{(a)}2_{\mathbf{k}_2}, 2_{-\mathbf{k}_2}\right\rangle \right) + \ldots\ldots \right) \times \\
&\quad \ldots\ldots\ldots
\end{aligned}
\tag{14}
$$

QUESTION: Shouldn't there be a factor of $n!$ before each state in the last three lines of (14)? Each creation operator leaves a factor of square root of (number of particles in ket plus 1).

Note if $\beta_k = 0$ for each value of $\mathbf{k}$, via (6.25), i.e., $\left|\alpha_k\right|^2 - \left|\beta_k\right|^2 = 1$, then $|\alpha_k| = 1$, and the b particle is the same as the a particle. Then (14) becomes

$$
\left|{}_{(b)}0\right\rangle = \prod_{\mathbf{k}} \left|{}_{(a)}0\right\rangle = \left|{}_{(a)}0_{\mathbf{k}_1,\mathbf{k}_{-1}}\right\rangle \left|{}_{(a)}0_{\mathbf{k}_2,\mathbf{k}_{-2}}\right\rangle \left|{}_{(a)}0_{\mathbf{k}_3,\mathbf{k}_{-3}}\right\rangle \ldots\ldots = \left|{}_{(a)}0\right\rangle \left|{}_{(a)}0\right\rangle \left|{}_{(a)}0\right\rangle \ldots\ldots \tag{15}
$$

I guess one can presume the RHS of (15) represents the a vacuum $\left|{}_{(a)}0\right\rangle$, which then equals the b vacuum.

6.5.1 The instantaneous lowest-energy state

The derivation of (6.34) is shown in Appendix A herein.

$$
\hat{H}(\eta) = \tfrac{1}{4}\int d^3\mathbf{k} \left[\underbrace{\left(\left(v^{*\prime}_k(\eta)\right)^2 + \omega_k^2 \right)}_{F_k^*} \hat{a}^-_{\mathbf{k}}\hat{a}^-_{-\mathbf{k}} + \underbrace{\left(\left(v'_k(\eta)\right)^2 + \omega_k^2 \right)}_{F_k} \hat{a}^+_{\mathbf{k}}\hat{a}^+_{-\mathbf{k}} + \underbrace{\left(\left|v'_k(\eta)\right|^2 + \omega_k^2 \left|v_k(\eta)\right|^2 \right)}_{E_k} \left(2\hat{a}^+_{\mathbf{k}}\hat{a}^-_{\mathbf{k}} + \delta^{(3)}(0)\right) \right] \quad (6.34) \text{ in M\&W}
$$

Note in elementary QFT, when H is not a function of time, we have (9). Using that value for v_k in M&W (6.35) and (6.36), we find

$$\left.\begin{array}{ll} E_k(\eta)\equiv\left|v_k'\right|^2+\omega_k^2\left|v_k\right|^2=\dfrac{2\omega_k^2}{\omega_k}=2\omega_k & (6.35) \\ F_k(\eta)\equiv\left(v_k'\right)^2+\omega_k^2\left(v_k\right)^2=0 & (6.36) \end{array}\right\} \quad a(t)=1 \text{ constant } \left(\text{no time dependence in } H\right) \tag{16}$$

With these values, the Hamiltonian (6.34) reduces to the familiar QFT form.

QUESTION: In (6.34) the terms

$$\hat{a}_{\mathbf{k}}^{-}\hat{a}_{-\mathbf{k}}^{-}F_k^{*}+\hat{a}_{\mathbf{k}}^{+}\hat{a}_{-\mathbf{k}}^{+}F_k \tag{17}$$

for non-zero F_k (when $H = H(t)$) appear like they would give rise to creation of a particle pair (with opposite direction **k**, so momentum would be conserved and no charge on each particle, since we are dealing with a real field) and destruction of a particle pair (with equal and opposite 3 momentum). But the expectation value of the terms in (17) would be zero, as the bra and ket left for each term after the operator action would not match.

Also, (6.36) is for the free Hamiltonian. For interactions, found after employing minimal substitution in the free Hamiltonian and the Dyson-Wicks expansion, some work would have to be done to see if this pair creation/destruction effect arises.

END QUESTION

Note that $\alpha_k = \alpha_k(\eta)$ used from just below (6.38) to (6.42) is *not* the α_k of the Bogolyubov relations used from (6.34) to (6.33).

<u>Minimizing $E_k(\eta_0)$ (6.38) to (6.42)</u>

The normalization relation (6.39) is good for any η, not just $\eta = \eta_0$.

Relations valid at any η

The relation after (6.39), with r_k and α_k real, is a guess at a workable solution (which will turn out correct) and is

$$v_k=r_k e^{i\alpha_k} \quad \text{where} \quad v_k=v_k(\eta) \qquad r_k=r_k(\eta) \qquad \alpha_k=\alpha_k(\eta), \tag{18}$$

which can be evaluated at $\eta = \eta_0$, but which should be valid at any η. Thus,

$$v_k'=\frac{dv_k(\eta)}{d\eta}=\frac{dr_k(\eta)}{d\eta}e^{i\alpha_k(\eta)}+i\alpha_k'(\eta)r_k(\eta)e^{i\alpha_k}=r_k'e^{i\alpha_k}+i\alpha_k'r_ke^{i\alpha_k}=r_k'e^{i\alpha_k}+i\alpha_k'v_k \tag{19}$$

Using (18) and (19) in (6.39) expressed in terms of η instead of η_0, we have

$$v_k'(\eta)v_k^{*}(\eta)-v_k(\eta)v_k^{*\prime}(\eta)=2i \qquad \text{Generally good at any } \eta \text{, not just } \eta_0 \text{ as in (6.39) of M\&W}$$

$$\underbrace{r_k'(\eta)e^{i\alpha_k(\eta)}r_k(\eta)e^{-i\alpha_k(\eta)}}_{\text{this cancels}}+i\alpha_k'(\eta)r_k(\eta)e^{i\alpha_k(\eta)}r_k(\eta)e^{-i\alpha_k(\eta)} \tag{20}$$

$$\overbrace{-r_k(\eta)e^{i\alpha_k(\eta)}r_k'(\eta)e^{-i\alpha_k(\eta)}}^{\text{with this}}+i\alpha_k'r_k(\eta)e^{i\alpha_k(\eta)}r_k(\eta)e^{-i\alpha_k(\eta)}=2i$$

or

$$\alpha_k'(\eta)r_k^2(\eta)=1 \qquad (6.40) \text{ in M\&W, but here good for any } \eta. \tag{21}$$

So, we use the LHS of (18) and the RHS of (19) in (6.35),

$$E_k(\eta)=\left|v_k'(\eta)\right|^2+\omega_k^2(\eta)\left|v_k(\eta)\right|^2 \qquad \text{(6.35) of M\&W}$$

$$=\left(r_k'(\eta)e^{i\alpha_k(\eta)}+i\alpha_k'(\eta)r_k(\eta)e^{i\alpha_k}\right)\left(r_k'(\eta)e^{-i\alpha_k(\eta)}-i\alpha_k'(\eta)r_k(\eta)e^{-i\alpha_k}\right)+\omega_k^2(\eta)\left(r_k(\eta)\right)^2 \tag{22}$$

$$=\left(r_k'(\eta)\right)^2+\left(\alpha_k'(\eta)r_k(\eta)\right)^2+\omega_k^2(\eta)\left(r_k(\eta)\right)^2 \qquad \text{1st part of 2nd row in (6.41) of M\&W, but here for any } \eta.$$

Using (21) in the second term in the last row of (22), we get

$$E_k(\eta)=\left(r_k'(\eta)\right)^2+\left(\frac{1}{r_k(\eta)}\right)^2+\omega_k^2(\eta)\left(r_k(\eta)\right)^2 \qquad \text{last part of (6.41) for any } \eta\left(\text{not just } \eta_0\right). \tag{23}$$

Relations for η_0 (η where E_k is minimum)

Now, we minimize (23) with respect to r_k and r'_k. We find the functional form of $r_k(\eta)$ in terms of $\omega_\kappa(\eta)$ that minimizes $E_k(\eta)$. We call the conformal time value at which that is true $\eta = \eta_0$. η_0 will be the symbol for the time at which vacuum energy is minimal.

Since $E_k(\eta)$ is a function of both r_k and r'_k we need to minimize it with respect to both, i.e.,

$$\frac{\partial E_k(\eta)}{\partial r_k}=0 \qquad \text{and} \qquad \frac{\partial E_k(\eta)}{\partial r_k'}=0. \tag{24}$$

Thus, from (23) and the LHS of (24),

$$0=\frac{\partial E_k(\eta)}{\partial r_k}=\frac{\partial}{\partial r_k}\left(\left(r_k'(\eta)\right)^2+\left(\frac{1}{r_k(\eta)}\right)^2+\omega_k^2(\eta)\left(r_k(\eta)\right)^2\right)=\underbrace{\frac{\partial}{\partial r_k}\left(r_k'(\eta)\right)^2}_{=0}+\frac{\partial}{\partial r_k}\left(\left(\frac{1}{r_k(\eta)}\right)^2+\omega_k^2(\eta)\left(r_k(\eta)\right)^2\right)$$
$$=2\frac{1}{r_k(\eta)}\frac{\partial}{\partial r_k}\frac{1}{r_k(\eta)}+2\omega_k^2(\eta)r_k(\eta)=-2\frac{1}{r_k(\eta)}\left(r_k(\eta)\right)^{-2}+2\omega_k^2(\eta)r_k(\eta), \tag{25}$$

or

$$\frac{1}{r_k(\eta)}\left(r_k(\eta)\right)^{-2}=\omega_k^2(\eta)r_k(\eta) \;\rightarrow\; \omega_k^2(\eta)r_k^4(\eta)=1 \;\rightarrow\; r_k(\eta)=\frac{1}{\sqrt{\omega_k(\eta)}}. \quad \text{True for min } E_k. \tag{26}$$

From (23) and the RHS of (24),

$$0=\frac{\partial E_k(\eta)}{\partial r_k'}=\frac{\partial}{\partial r_k'}\left(r_k'(\eta)\right)^2+0+0=2r_k'(\eta) \qquad \rightarrow \qquad r_k'(\eta)=0 \qquad \text{True for min } E_k. \tag{27}$$

Taking η_0 as the time at which E_k is a minimum, (26) and (27) are

$$r_k(\eta_0)=\frac{1}{\sqrt{\omega_k(\eta_0)}} \qquad\qquad r_k'(\eta_0)=0. \tag{28}$$

(28) being true minimizes vacuum energy with respect to the operators $\hat{a}_k^+,\hat{a}_k^-$ at η_0.

<u>Finding the form of $v_k(\eta_0)$</u>

From (21) (good for any η) and the LHS of (28) (good for η_0), we have

$$\alpha_k'(\eta_0)r_k^2(\eta_0)=1=\alpha_k'(\eta_0)\frac{1}{\omega_k(\eta_0)} \quad \rightarrow \quad \alpha_k'(\eta_0)=\omega_k(\eta_0). \tag{29}$$

From (18) (good for any η) and the LHS of (28) (good for η_0),, we have

$$v_k(\eta_0) = r_k(\eta_0)e^{i\alpha_k(\eta_0)} = \frac{1}{\sqrt{\omega_k(\eta_0)}}e^{i\alpha_k(\eta_0)} . \tag{30}$$

So from (19) (good for any η), (29), and (30), we have

$$v_k'(\eta_0) = r_k'(\eta_0)e^{i\alpha_k(\eta_0)} + i\alpha_k'(\eta_0)r_k(\eta_0)e^{i\alpha_k(\eta_0)} = r_k'(\eta_0)e^{i\alpha_k(\eta_0)} + i\omega_k(\eta_0)v_k(\eta_0) . \tag{31}$$

Note that (30) and (31) are true at η_0, the time at which vacuum energy $E_k(\eta) = E_k(\eta_0)$ is minimum.

Special Case: Constant ω_k for all times

If ω_k = constant (not a function of η), then from (26), r_k = constant for all time, and hence $r_k' = 0$ for all time. Since (26) and (27) are true for all time, we must have the same minimum E_k for all time.

In this case, (18) becomes

$$v_k(\eta) = \frac{1}{\sqrt{\omega_k}}e^{i\alpha_k(\eta)} \qquad \text{for } \omega_k \neq \omega_k(\eta) . \tag{32}$$

And (31) becomes

$$v_k'(\eta) = i\omega_k v_k(\eta) \qquad \text{for } \omega_k \neq \omega_k(\eta) . \tag{33}$$

And from (32) and (33), we must have

$$v_k(\eta) = \frac{1}{\sqrt{\omega_k}}e^{i\omega_k\eta} \qquad \text{for } \omega_k \neq \omega_k(\eta) , \tag{34}$$

which is the form we know so well from traditional QFT (which has constant ω_k and for which $\eta = t$).

General Case: ω_k a function of η at the special time η_0

If ω_k is a function of η, then at the time η_0 when $E_k(\eta)$ is a minimum = $E_k(\eta_0)$, (28) in (31) gives us

$$v_k'(\eta_0) = i\omega_k(\eta_0)v_k(\eta_0) \qquad \text{for } \omega_k = \omega_k(\eta), \text{ at minimum } E_k \text{ where } \eta = \eta_0 . \tag{35}$$

From (30),

$$v_k(\eta_0) = \frac{1}{\sqrt{\omega_k(\eta_0)}}e^{i\alpha_k(\eta_0)} \qquad \text{for } \omega_k = \omega_k(\eta), \text{ at minimum } E_k \text{ where } \eta = \eta_0 . \tag{36}$$

It can be insightful to compare (33) and (34) with (35) and (36). Note particularly, the exponential $i\omega_k\eta$ in (34) vs $i\alpha_k(\eta_0)$ in (36).

<u>(6.42) for constant H and non-constant H</u>

For the particular case where H is not a function of time, we can take $\eta = t$, and v_k is as found in (9), which satisfies the more general case relation (6.42).

Note that in the general case, for the initial conditions (at $\eta = \eta_0$), (6.42) into (6.36) yields $F_k(\eta_0) = 0$. F_k is zero initially, though in general it will not be so afterwards.

<u>No min vacuum energy for $\omega^2 < 0$</u>

For the paragraph after (6.42), note Fig. 2.

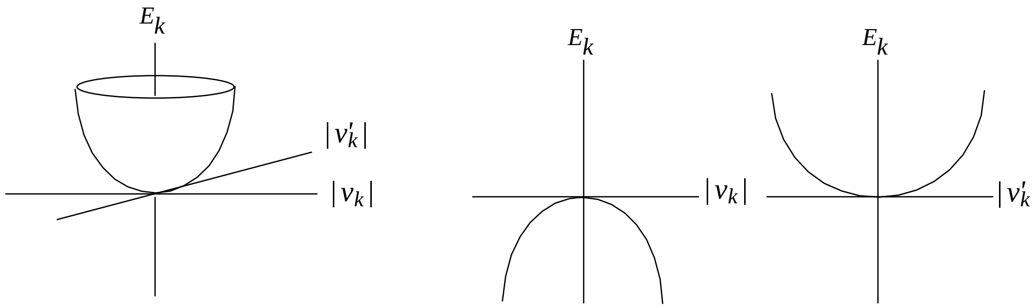

Figure 2. Plots of E_k of (6.35) for $\omega_{\mathbf{k}}^2$ Positive and Negative

Fig. 2 justifies the comment in M&W that there is no minimum E_k for negative ω_k^2.

<u>Hamiltonian diagonalization</u>

Note if $F_k = 0$, the Hamiltonian (6.34) becomes (6.43) and will not raise or lower a state containing *a* type particles. *a* type particle states are then eigenstates of the Hamiltonian, which they would not be for $F_k \neq 0$.

To see how this relates to "diagonalization", we can employ a vector space whose basis vectors are defined as shown in part in (37). Note that, for simplicity, we ignore the vacuum terms of form $\delta^3(0)E_k/4$.

$$\left|{}_v a_{k1}\right\rangle = \begin{bmatrix}1\\0\\0\\\vdots\\0\\\vdots\end{bmatrix} \quad \left|{}_v a_{-k1}\right\rangle = \begin{bmatrix}0\\1\\0\\\vdots\\0\\\vdots\end{bmatrix} \quad \left|{}_v a_{k2}\right\rangle = \begin{bmatrix}0\\0\\1\\\vdots\\0\\\vdots\end{bmatrix} \quad .... \quad \left|{}_v 2a_{k1}a_{-k1}\right\rangle = \begin{bmatrix}0\\0\\0\\\vdots\\1\\\vdots\end{bmatrix} \quad .... \tag{37}$$

In this space, the integrand of (6.43) is represented as a matrix, as in (38).

$$\text{Matrix in vector space for integrand of (6.43) without } \delta^{(3)}(0) \text{ term} \quad \hat{\mathscr{H}} = \begin{bmatrix}\omega_{k_1} &&&&& \\ & \omega_{k_1} &&&& \\ && \omega_{k_2} &&& \\ &&& \ddots && \\ &&&& 3\omega_{k_1} & \\ &&&&& \ddots\end{bmatrix} \quad \begin{matrix}F_k(\eta_0)=0\\ E_k(\eta_0)=2\omega_k\end{matrix} \tag{38).}$$

Thus, we have, for examples,

$$\begin{bmatrix}\omega_{k_1} &&&&& \\ & \omega_{k_1} &&&& \\ && \omega_{k_2} &&& \\ &&& \ddots && \\ &&&& 3\omega_{k_1} & \\ &&&&& \ddots\end{bmatrix}\begin{bmatrix}1\\0\\0\\\vdots\\0\\\vdots\end{bmatrix} = \omega_{k_1}\begin{bmatrix}1\\0\\0\\\vdots\\0\\\vdots\end{bmatrix} \quad \text{equivalent to} \quad \underbrace{\omega_{k_1}\hat{a}^{+}_{\mathbf{k}_1}\hat{a}^{-}_{\mathbf{k}_1}}_{(6.43)\text{ integrand}}\left|{}_v a_{k1}\right\rangle = \omega_{k_1}\left|{}_v a_{k1}\right\rangle \tag{39}$$

$$\begin{bmatrix} \omega_{k_1} & & & & & \\ & \omega_{k_1} & & & & \\ & & \omega_{k_2} & & & \\ & & & \ddots & & \\ & & & & 3\omega_{k_1} & \\ & & & & & \ddots \end{bmatrix}\begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 1 \\ \vdots \end{bmatrix} = 3\omega_{k_1}\begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 1 \\ \vdots \end{bmatrix} \quad \text{equivalent to} \quad \underbrace{\omega_{k_1}\hat{a}^{+}_{\mathbf{k}_1}\hat{a}^{-}_{\mathbf{k}_1} + \omega_{k_1}\hat{a}^{+}_{-\mathbf{k}_1}\hat{a}^{-}_{-\mathbf{k}_1}}_{\substack{(6.43)\text{ integral over all } \mathbf{k} \\ \text{leaves only these with} \\ \text{non-zero result}}}\left|{}_v 2a_{k1}a_{-k1}\right\rangle = 3\omega_{k_1}\left|{}_v 2a_{k1}a_{-k1}\right\rangle \tag{40}$$

Now consider the case where $F_k \neq 0$ (as would happen for $\eta > \eta_0$) where we then need to add extra terms from (6.34) into (6.43). Also, E_k would not have the simple form (as it does at η_0) of $2\omega_k$.

$$\underbrace{\left(\frac{E_k}{2}\hat{a}^{+}_{\mathbf{k}_1}\hat{a}^{-}_{\mathbf{k}_1} + \hat{a}^{-}_{\mathbf{k}_1}\hat{a}^{-}_{-\mathbf{k}_1}\frac{F^{*}_{k_1}}{4} + \hat{a}^{+}_{\mathbf{k}_1}\hat{a}^{+}_{-\mathbf{k}_1}\frac{F_{k_1}}{4}\right)}_{(6.43)\text{ integrand plus terms when } F_k \neq 0}\left|{}_v a_{k1}\right\rangle = \frac{E_k}{2}\left|{}_v a_{k1}\right\rangle + 0 + \frac{F_{k_1}}{4}\sqrt{2}\left|{}_v 2a_{k1}a_{-k1}\right\rangle \qquad \begin{matrix} F_k(\eta) \neq 0 \\ E_k(\eta) \neq 2\omega_k \end{matrix} \tag{41}$$

Expressing (41) in vector space form, we have

$$\begin{bmatrix} \frac{E_{k_1}}{2} & & & & & \\ & \frac{E_{k_1}}{2} & & & & \\ & & \frac{E_{k_2}}{2} & & & \\ & & & \ddots & & \\ \frac{F_{k_1}}{4}\sqrt{2} & & & & 3\frac{E_{k_1}}{2} & \\ & & & & & \ddots \end{bmatrix}\begin{bmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \\ \vdots \end{bmatrix} = \begin{bmatrix} \frac{E_{k_1}}{2} \\ 0 \\ 0 \\ \vdots \\ \frac{F_{k_1}}{4}\sqrt{2} \\ \vdots \end{bmatrix} \quad (\text{many terms not shown}) \qquad \text{equivalent to (41)} \tag{42}$$

The result of the operator (matrix in vector space) operation on the ket of (41) [vector of (42)] is not the same ket (vector) we started with. So the ket (vector) is not an eigenstate of the H operator (matrix). Anytime we have off diagonal terms in a matrix in a vector space, the basis vectors of that space will not be eigenvectors of that matrix.

So, this is what is meant by $F_k \neq 0$ making the Hamiltonian non-diagonal (in vector space). Note that adding in the vacuum terms $\delta^3(0)E_k/4 = \delta^3(0)\omega_k/2$ to the matrix of (38) to (40) would leave it in diagonal form.

<u>Vacuum state an eigenstate of H only if $F_k = 0$ for all η</u> (top line of pg. 74)

Note in (6.34) that if $F_k \neq 0$, we have the Hamiltonian acting on the vacuum where

$$\hat{H}\left|0\right\rangle \quad \text{has a term} \quad F_k\,\hat{a}^{+}_{\mathbf{k}}\hat{a}^{+}_{-\mathbf{k}}\left|0\right\rangle = F_k\left|\phi_{\mathbf{k}},\phi_{-\mathbf{k}}\right\rangle \quad \text{so} \quad \hat{H}\left|0\right\rangle \neq (\text{constant})\left|0\right\rangle. \tag{43}$$

Hence, in this case the vacuum is not an eigenstate of the Hamiltonian operator. If, on the other hand, $F_k = 0$, the only terms left in (6.34) are number operators and constants, so operation by $\hat{H}$ on $|0\rangle$ will give a constant times $|0\rangle$, and thus $|0\rangle$ is an eigenstate of $\hat{H}$.

<u>Different vacuums (lowest energy states) at different η and Bogolyubov transformations</u> (upper part of pg. 74)

For ω_k as a function of time, we have different mode functions $v_k(\eta)$ (different basis states) at different times. $v_k(\eta_0)$ is not the same as $v_k(\eta_1)$, for given **k**. However, either set of basis mode functions spans the space of all mode functions. At each **k**, the pair of mode functions $v_k(\eta_0)$ and $v_k^*(\eta_0)$ spans the space of mode functions of **k**, and so does the pair $v_k(\eta_1)$ and $v_k^*(\eta_1)$. We can express any general mode function in terms of either. So, we can choose our general mode function as the specific one $v_k(\eta_0)$ and express it in terms of $v_k(\eta_1)$ and $v_k^*(\eta_1)$. That is,

$$\underbrace{v_k(\eta_0)}_{\substack{\hat{b}^{\pm}_{\mathbf{k}}\text{ type}\\ \text{opers}}} = \alpha_k \underbrace{v_k(\eta_1)}_{\substack{\hat{a}^{\pm}_{\mathbf{k}}\text{ type}\\ \text{opers}}} + \beta_k \underbrace{v_k^*(\eta_1)}_{\substack{\hat{a}^{\pm}_{\mathbf{k}}\text{ type}\\ \text{opers}}}, \tag{44}$$

which parallels (6.24) on pg. 68

$$\underbrace{u_k(\eta)}_{\substack{\hat{b}^{\pm}_{\mathbf{k}}\text{ type}\\ \text{opers}}} = \alpha_k \underbrace{v_k(\eta)}_{\substack{\hat{a}^{\pm}_{\mathbf{k}}\text{ type}\\ \text{opers}}} + \beta_k \underbrace{v_k^*(\eta)}_{\substack{\hat{a}^{\pm}_{\mathbf{k}}\text{ type}\\ \text{opers}}} \qquad \text{M\&W (6.24),}$$

Thus, the relation of (6.32), pg. 70, derived by M&W for a and b type operator fields of (6.24) using the Bogolyubov transformations,

$$\left\langle {}_{(b)}0 \left| \hat{N}^{(a)}_{\mathbf{k}} \right| {}_{(b)}0 \right\rangle = \left\langle {}_{(b)}0 \left| \hat{a}^{+}_{\mathbf{k}} \hat{a}^{-}_{\mathbf{k}} \right| {}_{(b)}0 \right\rangle = \left|\beta_k\right|^2 \delta^{(3)}(0) \qquad \text{M\&W (6.32),}$$

corresponds directly for our case of (44) to

$$\left\langle {}_{\eta_0}0 \left| \hat{N}^{\eta_1}_{\mathbf{k}} \right| {}_{\eta_0}0 \right\rangle = \left\langle {}_{\eta_0}0 \left| \hat{a}^{+}_{\mathbf{k}}(\eta_1) \hat{a}^{-}_{\mathbf{k}}(\eta_1) \right| {}_{\eta_0}0 \right\rangle = \left|\beta_k\right|^2 \delta^{(3)}(0). \tag{45}$$

Using (45) in our Hamiltonian $\hat{H}(\eta_1)$, we find the relation just above mid page on pg. 74,

$$\begin{aligned}\left\langle {}_{\eta_0}0 \left| \hat{H}(\eta_1) \right| {}_{\eta_0}0 \right\rangle &= \left\langle {}_{\eta_0}0 \left| \int d^3\mathbf{x}\, \omega_k(\eta_1) \left(\hat{N}^{\eta_1}_{\mathbf{k}} + \tfrac{1}{2}\delta^{(3)}(0) \right) \right| {}_{\eta_0}0 \right\rangle \\ &= \left\langle {}_{\eta_0}0 \left| \int d^3\mathbf{x}\, \omega_k(\eta_1) \left(\hat{a}^{+}_{\mathbf{k}}(\eta_1) \hat{a}^{-}_{\mathbf{k}}(\eta_1) + \tfrac{1}{2}\delta^{(3)}(0) \right) \right| {}_{\eta_0}0 \right\rangle \qquad \text{M\&W mid pg. 74} \\ &= \delta^{(3)}(0) \int d^3\mathbf{x}\, \omega_k(\eta_1) \left(\left|\beta_k\right|^2 + \tfrac{1}{2} \right).\end{aligned} \tag{46}$$

M&W Remark: minimal fluctuations (mid pg. 74)

<u>Argument in M&W</u>

The argument is made here that because of the Wronskian normalization (11) of setting (6.14) to $2i$, we get a kind of uncertainty principle relation between the particle field and its time derivative (related to conjugate momentum of the field). The larger v_k, the smaller v_k'; and vice versa.

<u>Counter argument by Klauber</u>

Fermion example

However, for fermions, the RQM probability density (see Klauber (4-34) and (4-35), pg 92) has different form than that of (11), and is

$$\rho = \bar{\psi}\gamma^0\psi = \psi^\dagger \underbrace{\gamma^0\gamma^0}_{I} \psi = \underbrace{\psi^\dagger\psi = -i\pi\psi}_{\text{from (4-62) in Klauber}} \qquad \text{where conjugate momentum } \pi = i\psi^\dagger. \tag{47}$$

(47) means we need a normalization for which ρ integrated over all space would give unity for a single particle. This leads to a normalization of the time dependent factors in ψ, comparable to the relation (11) for scalars, of

$$v_k^\dagger(\eta)\, v_k(\eta) = 1. \tag{48}$$

We might conclude, via uncertainty principle logic, that a larger v_k means a smaller $v_k^\dagger$. But the latter is just the complex conjugate transpose of the former, so both must have the same magnitude. One can't get larger while the other gets smaller.

Thus, it seems the arguments of this remark regarding minimal fluctuations cannot hold for fermions. That brings into question the elevation of it to a general principle (which the Heisenberg principle is).

Scalar example

Additionally, for scalars in non time dependent QFT, $\phi_{\mathbf{k}}$ and its conjugate momentum have forms (where $b(\mathbf{k})$ corresponds here to antiparticle fields)

$$\phi(x) = \sum_{\mathbf{k}} \frac{1}{\sqrt{2V\omega_{\mathbf{k}}}} \left(a(\mathbf{k}) e^{-ikx} + b^\dagger(\mathbf{k}) e^{ikx} \right) \qquad \pi(x) = \dot{\phi}^\dagger(x) = \sum_{\mathbf{k}} \frac{i\omega_{\mathbf{k}}}{\sqrt{2V\omega_{\mathbf{k}}}} \left(-a(\mathbf{k}) e^{-ikx} + b^\dagger(\mathbf{k}) e^{ikx} \right). \tag{49}$$

So in any factor of $\phi\dot{\phi}^\dagger$, such as that giving rise to (11), for given $\mathbf{k}$, we get factors of

$$\phi_{\mathbf{k}}(x)\dot{\phi}_{\mathbf{k}}^\dagger(x) \propto \frac{1}{\sqrt{2V\omega_{\mathbf{k}}}} \frac{i\omega_{\mathbf{k}}}{\sqrt{2V\omega_{\mathbf{k}}}} = \frac{i}{2V} = \text{ a constant for given } V\,. \tag{50}$$

If we arbitrarily made the magnitude of $\phi_{\mathbf{k}}$ larger by a factor of K, then from (49), $\dot{\phi}_{\mathbf{k}}^\dagger$ would be larger by the same factor, and (50) larger by K^2. And then the integration of (11) over V would not be 1, but K^2. Additionally, the canonical commutation relations (4.13) would yield coefficient commutation relations [see (4.24) and (6.23)]

$$\left[\hat{a}_{\mathbf{k}}^-, \hat{a}_{\mathbf{k}'}^+ \right] = K^2\, \delta_{\mathbf{k}\mathbf{k}'} \;\; (\text{discrete}) \qquad\qquad \left[\hat{a}_{\mathbf{k}}^-, \hat{a}_{\mathbf{k}'}^+ \right] = K^2\, \delta(\mathbf{k}-\mathbf{k}') \quad (\text{continuous}) \;, \tag{51}$$

which would mess up the theory for $K^2 \neq 1$.

Conclusion:

The uncertainty principle seems applicable to wave packets, made of a superposition of many $\mathbf{k}$ states, not to a particle (or fields) made of a single state $\mathbf{k}$ (hypothetical though such a state would be in the real world where all particles are really wave packets.)

End counter argument

6.5.2 Ambiguity of the vacuum state

Mid pg 75 comment about contradicting normalization if $F_k = 0$

The condition $F_k(\eta_0) = 0$ means [see (43)] the vacuum is an eigenstate of the Hamiltonian. If, in addition to that, $\omega_k^2 < 0$, we have [see (6.36) or (16) herein]

$$F_k(\eta) \equiv (v_k')^2 + \omega_k^2 (v_k)^2 = (v_k')^2 - \left|\omega_k^2\right| (v_k)^2 = 0 \qquad\qquad (6.36) \text{ set equal to } 0\,, \tag{52}$$

The solution to (52) has the form of an exponential raised to a real, not imaginary, power,

$$v_k(\eta) = (\text{constant})\, e^{\pm\int \left|\omega_k^2\right| d\eta} \xrightarrow[\text{not a function of time}]{\text{for special case where } \omega_k} v_k(\eta) = (\text{constant})\, e^{\pm\left|\omega_k^2\right|\eta}. \tag{53}$$

For this case, v_k is real, so substitution of (53) into the normalization condition (10) [(6.14) set to $2i$] gives us

$$v_k' v_k^* - v_k v_k^{*\prime} = 2i \quad \xrightarrow{v_k \text{ real}} \quad v_k' v_k - v_k v_k' = 0\,. \tag{54}$$

Both sides of (54) can't be true, so we can't define a real v_k that leads to a viable theory. If we have no viable theory, we can't define a lowest energy eigenstate associated with the v_k.

Mid pg 75 comment on accelerated detector

It is not widely appreciated, but accelerated frames are NOT curved. They can be formalized as a transformation from an inertial frame under the transformation embodying the acceleration. But the Riemann curvature tensor, like any tensor, if zero in one frame, is zero in any frame it can be transformed into. So the Riemann tensor in an accelerated frame is zero, i..e, the 4D frame is flat.

This is not true of gravity. You cannot transform, everywhere, a Minkowski frame to a frame with gravity arising from the presence of matter.

Geodesics in both accelerated and gravitation frames (all non-inertial frames) are curved. 4D spacetime itself is flat in accelerated frames and curved in gravitation ones.

6.7 An example of particle production

Particle number density (pg. 82)

The derivation of the RHS of (6.61) is shown below. The absolute value of the sin function is not shown as that is trivial.

$$\begin{aligned}|\beta_k|^2 \quad \rightarrow \quad &\left(\tfrac{1}{2}\right)^2\left(\frac{\Omega}{\omega}-\frac{\omega}{\Omega}\right)^2=\frac{1}{4}\left(\frac{\Omega^2}{\omega^2}+\frac{\omega^2}{\Omega^2}-2\right)=\frac{1}{4}\left(\frac{k^2-m_0^2}{k^2+m_0^2}+\frac{k^2+m_0^2}{k^2-m_0^2}-2\right)\\ &=\frac{1}{4}\left(\frac{k^4+m_0^4-2k^2m_0^2k^4+k^4+m_0^4+2k^2m_0^2-2k^4+2m_0^4}{k^4-m_0^4}\right)=\frac{1}{4}\left(\frac{4m_0^4}{k^4-m_0^4}\right)=\frac{m_0^4}{k^4-m_0^4}\ .\end{aligned} \tag{55}$$

Appendix A. Derivation of M&W (6.34)

The following parallels the derivation of Klauber in his Sect. 3.4.1, pgs 53-53 for the free Hamiltonian operator in terms of coefficient operators. We use M&W notation here, not Klauber's. M&W is for continuous solutions, whereas the treatment in Klauber is for the discrete solutions. Also, Klauber works in a Minkowski spacetime with constant *m*, whereas M&W work in an expanding spacetime encapusulated in $m_{eff}(\eta)$. Also, M&W work with a real field $\hat{\chi}$, whereas ϕ in Klauber was complex.

6.5.1 The instantaneous lowest-energy state (Section starts on pg. 71 of M&W)

At the top of pg. 72, M&W find the Hamiltonian operator in an expanding universe $\hat{H}(\eta)$ in terms of the operators and mode functions by substituting (6.20) of pg. 67 into (6.19) of pg. 67.

Those starting relations, plus the commutation relations (which we will use) are

$$\hat{H}(\eta)=\frac{1}{2}\int d^3\mathbf{x}\left[\hat{\pi}^2+(\nabla\hat{\chi})^2+m_{\mathrm{eff}}^2(\eta)\,\hat{\chi}^2\right] \tag{6.19}$$

$$\hat{\chi}(\mathbf{x},\eta)=\frac{1}{\sqrt{2}}\frac{1}{(2\pi)^{3/2}}\int d^3\mathbf{k}\left(e^{i\mathbf{k}\bullet\mathbf{x}}v_k^*(\eta)\,\hat{a}_{\mathbf{k}}^-+e^{-i\mathbf{k}\bullet\mathbf{x}}v_k(\eta)\,\hat{a}_{\mathbf{k}}^+\right). \tag{6.20}$$

$$\left[\hat{a}_{\mathbf{k}}^-,\hat{a}_{\underset{\sim}{\mathbf{k}}}^+\right]=i\delta\left(\mathbf{k}-\underset{\sim}{\mathbf{k}}\right), \tag{6.23}$$

where we use the dummy variable $\underset{\sim}{\mathbf{k}}$ in place of M&W's dummy variable $\mathbf{k}'$, as the prime is used elsewhere to designate derivation with respect to η.

The conjugate momentum is

$$\hat{\pi}(\mathbf{x},\eta)=\hat{\chi}'(\mathbf{x},\eta)=\frac{1}{\sqrt{2}}\frac{1}{(2\pi)^{3/2}}\int d^3\mathbf{k}\left(e^{i\mathbf{k}\bullet\mathbf{x}}v_k^{*\prime}(\eta)\,\hat{a}_{\mathbf{k}}^-+e^{-i\mathbf{k}\bullet\mathbf{x}}v_k'(\eta)\,\hat{a}_{\mathbf{k}}^+\right). \tag{56}$$

<u>The first term in (6.19)</u> becomes

$$\hat{H}(\eta)_{\text{1st term}}=\frac{1}{2}\int d^3\mathbf{x}\;\hat{\pi}^2=\frac{1}{2}\int d^3\mathbf{x}\begin{pmatrix}\frac{1}{\sqrt{2}}\frac{1}{(2\pi)^{3/2}}\int d^3\mathbf{k}\left(e^{i\mathbf{k}\bullet\mathbf{x}}v_k^{*\prime}(\eta)\,\hat{a}_{\mathbf{k}}^-+e^{-i\mathbf{k}\bullet\mathbf{x}}v_k'(\eta)\,\hat{a}_{\mathbf{k}}^+\right)\times\\ \frac{1}{\sqrt{2}}\frac{1}{(2\pi)^{3/2}}\int d^3\underset{\sim}{\mathbf{k}}\left(e^{i\underset{\sim}{\mathbf{k}}\bullet\mathbf{x}}v_{\underset{\sim}{k}}^{*\prime}(\eta)\,\hat{a}_{\underset{\sim}{\mathbf{k}}}^-+e^{-i\underset{\sim}{\mathbf{k}}\bullet\mathbf{x}}v_{\underset{\sim}{k}}'(\eta)\,\hat{a}_{\underset{\sim}{\mathbf{k}}}^+\right)\end{pmatrix} \tag{57}$$

$$=\frac{1}{4}\frac{1}{(2\pi)^3}\int d^3\mathbf{x}\int d^3\mathbf{k}\int d^3\underset{\sim}{\mathbf{k}}\begin{pmatrix}e^{i(\mathbf{k}+\underset{\sim}{\mathbf{k}})\bullet\mathbf{x}}v_k^{*\prime}(\eta)\,v_{\underset{\sim}{k}}^{*\prime}(\eta)\,\hat{a}_{\mathbf{k}}^-\hat{a}_{\underset{\sim}{\mathbf{k}}}^-+e^{i(\mathbf{k}-\underset{\sim}{\mathbf{k}})\bullet\mathbf{x}}v_k^{*\prime}(\eta)\,v_{\underset{\sim}{k}}'(\eta)\,\hat{a}_{\mathbf{k}}^-\hat{a}_{\underset{\sim}{\mathbf{k}}}^+\\ +e^{-i(\mathbf{k}-\underset{\sim}{\mathbf{k}})\bullet\mathbf{x}}v_k'(\eta)\,v_{\underset{\sim}{k}}^{*\prime}(\eta)\,\hat{a}_{\mathbf{k}}^+\hat{a}_{\underset{\sim}{\mathbf{k}}}^-+e^{-i(\mathbf{k}+\underset{\sim}{\mathbf{k}})\bullet\mathbf{x}}v_{\underset{\sim}{k}}'(\eta)\,v_{\underset{\sim}{k}}'(\eta)\,\hat{a}_{\mathbf{k}}^+\hat{a}_{\underset{\sim}{\mathbf{k}}}^+\end{pmatrix} \tag{58}$$

Since the Dirac delta function is

$$\delta^{(3)}\left(\mathbf{k}-\underset{\sim}{\mathbf{k}}\right)=\frac{1}{(2\pi)^3}\int d^3\mathbf{x}\, e^{i\left(\mathbf{k}-\underset{\sim}{\mathbf{k}}\right)}, \tag{59}$$

the integral over **x** in (58) gives us

$$\begin{aligned}\hat{H}(\eta)_{1\text{st term}} &= \frac{1}{4}\int d^3\mathbf{k}\int d^3\underset{\sim}{\mathbf{k}}\begin{pmatrix}\delta\left(\mathbf{k}+\underset{\sim}{\mathbf{k}}\right)v_k^{*\prime}(\eta)\,v_{\underset{\sim}{k}}^{*\prime}(\eta)\,\hat{a}_{\mathbf{k}}^{-}\hat{a}_{\underset{\sim}{\mathbf{k}}}^{-}+\delta\left(\mathbf{k}-\underset{\sim}{\mathbf{k}}\right)v_k^{*\prime}(\eta)\,v_{\underset{\sim}{k}}^{\prime}(\eta)\,\hat{a}_{\mathbf{k}}^{-}\hat{a}_{\underset{\sim}{\mathbf{k}}}^{+}\\ +\delta\left(\mathbf{k}-\underset{\sim}{\mathbf{k}}\right)v_k^{\prime}(\eta)\,v_{\underset{\sim}{k}}^{*\prime}(\eta)\,\hat{a}_{\mathbf{k}}^{+}\hat{a}_{\underset{\sim}{\mathbf{k}}}^{-}+\delta\left(\mathbf{k}+\underset{\sim}{\mathbf{k}}\right)v_{\underset{\sim}{k}}^{\prime}(\eta)\,v_{\underset{\sim}{k}}^{\prime}(\eta)\,\hat{a}_{\mathbf{k}}^{+}\hat{a}_{\underset{\sim}{\mathbf{k}}}^{+}\end{pmatrix}\\ &= \frac{1}{4}\int d^3\mathbf{k}\begin{pmatrix}v_k^{*\prime}(\eta)\,v_k^{*\prime}(\eta)\,\hat{a}_{\mathbf{k}}^{-}\hat{a}_{-\mathbf{k}}^{-}+v_k^{*\prime}(\eta)\,v_k^{\prime}(\eta)\,\hat{a}_{\mathbf{k}}^{-}\hat{a}_{\mathbf{k}}^{+}\\ +v_k^{\prime}(\eta)\,v_k^{*\prime}(\eta)\,\hat{a}_{\mathbf{k}}^{+}\hat{a}_{\mathbf{k}}^{-}+v_k^{\prime}(\eta)\,v_k^{\prime}(\eta)\,\hat{a}_{\mathbf{k}}^{+}\hat{a}_{-\mathbf{k}}^{+}\end{pmatrix}\\ &= \frac{1}{4}\int d^3\mathbf{k}\begin{pmatrix}v_k^{*\prime}(\eta)\,v_k^{*\prime}(\eta)\,\hat{a}_{\mathbf{k}}^{-}\hat{a}_{-\mathbf{k}}^{-}+v_k^{*\prime}(\eta)\,v_k^{\prime}(\eta)\left(\hat{a}_{\mathbf{k}}^{-}\hat{a}_{\mathbf{k}}^{+}+\hat{a}_{\mathbf{k}}^{+}\hat{a}_{\mathbf{k}}^{-}\right)\\ +v_k^{\prime}(\eta)\,v_k^{\prime}(\eta)\,\hat{a}_{\mathbf{k}}^{+}\hat{a}_{-\mathbf{k}}^{+}\end{pmatrix}\end{aligned} \tag{60}$$

$$=\frac{1}{4}\int d^3\mathbf{k}\left(v_k^{*\prime}(\eta)\,v_k^{*\prime}(\eta)\,\hat{a}_{\mathbf{k}}^{-}\hat{a}_{-\mathbf{k}}^{-}+v_k^{*\prime}(\eta)\,v_k^{\prime}(\eta)\left(\hat{a}_{\mathbf{k}}^{-}\hat{a}_{\mathbf{k}}^{+}+\hat{a}_{\mathbf{k}}^{+}\hat{a}_{\mathbf{k}}^{-}\right)+v_k^{\prime}(\eta)\,v_k^{\prime}(\eta)\,\hat{a}_{\mathbf{k}}^{+}\hat{a}_{-\mathbf{k}}^{+}\right) \tag{61}$$

With (6.23), we have

$$\hat{H}(\eta)_{1\text{st term}}=\frac{1}{4}\int d^3\mathbf{k}\left(\left(v_k^{*\prime}(\eta)\right)^2\hat{a}_{\mathbf{k}}^{-}\hat{a}_{-\mathbf{k}}^{-}+\left|v_k^{\prime}(\eta)\right|^2\left(2\hat{a}_{\mathbf{k}}^{+}\hat{a}_{\mathbf{k}}^{-}+\delta^{(3)}(0)\right)+\left(v_k^{\prime}(\eta)\right)^2\hat{a}_{\mathbf{k}}^{+}\hat{a}_{-\mathbf{k}}^{+}\right). \tag{62}$$

<u>The second term in (6.19)</u> becomes

$$\hat{H}(\eta)_{2\text{nd term}}=\frac{1}{2}\int d^3\mathbf{x}\,\left(\nabla\hat{\chi}\right)^2=\frac{1}{2}\int d^3\mathbf{x}\begin{pmatrix}\frac{1}{\sqrt{2}}\frac{1}{(2\pi)^{3/2}}\int d^3\mathbf{k}\left(i\mathbf{k}e^{i\mathbf{k}\bullet\mathbf{x}}v_k^{*}(\eta)\,\hat{a}_{\mathbf{k}}^{-}-i\mathbf{k}e^{-i\mathbf{k}\bullet\mathbf{x}}v_k(\eta)\,\hat{a}_{\mathbf{k}}^{+}\right)\times\\ \frac{1}{\sqrt{2}}\frac{1}{(2\pi)^{3/2}}\int d^3\underset{\sim}{\mathbf{k}}\left(i\underset{\sim}{\mathbf{k}}e^{i\underset{\sim}{\mathbf{k}}\bullet\mathbf{x}}v_{\underset{\sim}{k}}^{*}(\eta)\,\hat{a}_{\underset{\sim}{\mathbf{k}}}^{-}-i\underset{\sim}{\mathbf{k}}e^{-i\underset{\sim}{\mathbf{k}}\bullet\mathbf{x}}v_{\underset{\sim}{k}}(\eta)\,\hat{a}_{\underset{\sim}{\mathbf{k}}}^{+}\right)\end{pmatrix} \tag{63}$$

$$=\frac{1}{4}\frac{1}{(2\pi)^3}\int d^3\mathbf{x}\int d^3\mathbf{k}\int d^3\underset{\sim}{\mathbf{k}}\begin{pmatrix}-\mathbf{k}\bullet\underset{\sim}{\mathbf{k}}e^{i\left(\mathbf{k}+\underset{\sim}{\mathbf{k}}\right)\bullet\mathbf{x}}v_k^{*}(\eta)\,v_{\underset{\sim}{k}}^{*}(\eta)\,\hat{a}_{\mathbf{k}}^{-}\hat{a}_{\underset{\sim}{\mathbf{k}}}^{-}+\mathbf{k}\bullet\underset{\sim}{\mathbf{k}}e^{i\left(\mathbf{k}-\underset{\sim}{\mathbf{k}}\right)\bullet\mathbf{x}}v_k^{*}(\eta)\,v_{\underset{\sim}{k}}(\eta)\,\hat{a}_{\mathbf{k}}^{-}\hat{a}_{\underset{\sim}{\mathbf{k}}}^{+}\\ +\mathbf{k}\bullet\underset{\sim}{\mathbf{k}}e^{-i\left(\mathbf{k}-\underset{\sim}{\mathbf{k}}\right)\bullet\mathbf{x}}v_k(\eta)\,v_{\underset{\sim}{k}}^{*}(\eta)\,\hat{a}_{\mathbf{k}}^{+}\hat{a}_{\underset{\sim}{\mathbf{k}}}^{-}-\mathbf{k}\bullet\underset{\sim}{\mathbf{k}}e^{-i\left(\mathbf{k}+\underset{\sim}{\mathbf{k}}\right)\bullet\mathbf{x}}v_{\underset{\sim}{k}}(\eta)\,v_{\underset{\sim}{k}}(\eta)\,\hat{a}_{\mathbf{k}}^{+}\hat{a}_{\underset{\sim}{\mathbf{k}}}^{+}\end{pmatrix} \tag{64}$$

Using (59), this becomes

$$\begin{aligned}&=\frac{1}{4}\int d^3\mathbf{k}\int d^3\underset{\sim}{\mathbf{k}}\begin{pmatrix}-\mathbf{k}\bullet\underset{\sim}{\mathbf{k}}\delta\left(\mathbf{k}+\underset{\sim}{\mathbf{k}}\right)v_k^{*}(\eta)\,v_{\underset{\sim}{k}}^{*}(\eta)\,\hat{a}_{\mathbf{k}}^{-}\hat{a}_{\underset{\sim}{\mathbf{k}}}^{-}+\mathbf{k}\bullet\underset{\sim}{\mathbf{k}}\delta\left(\mathbf{k}-\underset{\sim}{\mathbf{k}}\right)v_k^{*}(\eta)\,v_{\underset{\sim}{k}}(\eta)\,\hat{a}_{\mathbf{k}}^{-}\hat{a}_{\underset{\sim}{\mathbf{k}}}^{+}\\ +\mathbf{k}\bullet\underset{\sim}{\mathbf{k}}\delta\left(\mathbf{k}-\underset{\sim}{\mathbf{k}}\right)v_k(\eta)\,v_{\underset{\sim}{k}}^{*}(\eta)\,\hat{a}_{\mathbf{k}}^{+}\hat{a}_{\underset{\sim}{\mathbf{k}}}^{-}-\mathbf{k}\bullet\underset{\sim}{\mathbf{k}}\delta\left(\mathbf{k}+\underset{\sim}{\mathbf{k}}\right)v_{\underset{\sim}{k}}(\eta)\,v_{\underset{\sim}{k}}(\eta)\,\hat{a}_{\mathbf{k}}^{+}\hat{a}_{\underset{\sim}{\mathbf{k}}}^{+}\end{pmatrix}\\ &=\frac{1}{4}\int d^3\mathbf{k}\begin{pmatrix}\mathbf{k}^2v_k^{*}(\eta)\,v_k^{*}(\eta)\,\hat{a}_{\mathbf{k}}^{-}\hat{a}_{-\mathbf{k}}^{-}+\mathbf{k}^2v_k^{*}(\eta)\,v_k(\eta)\,\hat{a}_{\mathbf{k}}^{-}\hat{a}_{\mathbf{k}}^{+}\\ +\mathbf{k}^2v_k(\eta)\,v_k^{*}(\eta)\,\hat{a}_{\mathbf{k}}^{+}\hat{a}_{\mathbf{k}}^{-}+\mathbf{k}^2v_k(\eta)\,v_k(\eta)\,\hat{a}_{\mathbf{k}}^{+}\hat{a}_{-\mathbf{k}}^{+}\end{pmatrix}\\ &=\frac{1}{4}\int d^3\mathbf{k}\ \mathbf{k}^2\left(\left(v_k^{*}(\eta)\right)^2\hat{a}_{\mathbf{k}}^{-}\hat{a}_{-\mathbf{k}}^{-}+v_k^{*}(\eta)\,v_k(\eta)\left(\hat{a}_{\mathbf{k}}^{-}\hat{a}_{\mathbf{k}}^{+}+\hat{a}_{\mathbf{k}}^{+}\hat{a}_{\mathbf{k}}^{-}\right)+\left(v_k(\eta)\right)^2\hat{a}_{\mathbf{k}}^{+}\hat{a}_{-\mathbf{k}}^{+}\right)\end{aligned} \tag{65}$$

With (6.23), we have

$$\hat{H}(\eta)_{2\text{nd term}}=\frac{1}{4}\int d^3\mathbf{k}\ \mathbf{k}^2\left(\left(v_k^{*}(\eta)\right)^2\hat{a}_{\mathbf{k}}^{-}\hat{a}_{-\mathbf{k}}^{-}+\left|v_k(\eta)\right|^2\left(2\hat{a}_{\mathbf{k}}^{+}\hat{a}_{\mathbf{k}}^{-}+\delta^{(3)}(0)\right)+\left(v_k(\eta)\right)^2\hat{a}_{\mathbf{k}}^{+}\hat{a}_{-\mathbf{k}}^{+}\right). \tag{66}$$

The third term in (6.19) becomes

$$\hat{H}(\eta)_{3\text{rd term}} = \frac{1}{2}\int d^3\mathbf{x}\, m_{\text{eff}}^2(\eta)\,\hat{\chi}^2$$

$$= \frac{1}{2}\int d^3\mathbf{x}\, m_{\text{eff}}^2(\eta)\begin{pmatrix} \frac{1}{\sqrt{2}}\frac{1}{(2\pi)^{3/2}}\int d^3\mathbf{k}\left(e^{i\mathbf{k}\bullet\mathbf{x}}v_k^*(\eta)\hat{a}_{\mathbf{k}}^- + e^{-i\mathbf{k}\bullet\mathbf{x}}v_k(\eta)\hat{a}_{\mathbf{k}}^+\right)\times \\ \frac{1}{\sqrt{2}}\frac{1}{(2\pi)^{3/2}}\int d^3\underset{\sim}{\mathbf{k}}\left(e^{i\underset{\sim}{\mathbf{k}}\bullet\mathbf{x}}v_{\underset{\sim}{k}}^*(\eta)\hat{a}_{\underset{\sim}{\mathbf{k}}}^- + e^{-i\underset{\sim}{\mathbf{k}}\bullet\mathbf{x}}v_{\underset{\sim}{k}}(\eta)\hat{a}_{\underset{\sim}{\mathbf{k}}}^+\right) \end{pmatrix} \tag{67}$$

$$= \frac{1}{4}\frac{1}{(2\pi)^3}\int d^3\mathbf{x}\int d^3\mathbf{k}\int d^3\underset{\sim}{\mathbf{k}}\, m_{\text{eff}}^2(\eta)\begin{pmatrix} e^{i(\mathbf{k}+\underset{\sim}{\mathbf{k}})\bullet\mathbf{x}}v_k^*(\eta)v_{\underset{\sim}{k}}^*(\eta)\hat{a}_{\mathbf{k}}^-\hat{a}_{\underset{\sim}{\mathbf{k}}}^- + e^{i(\mathbf{k}-\underset{\sim}{\mathbf{k}})\bullet\mathbf{x}}v_k^*(\eta)v_{\underset{\sim}{k}}(\eta)\hat{a}_{\mathbf{k}}^-\hat{a}_{\underset{\sim}{\mathbf{k}}}^+ \\ + e^{-i(\mathbf{k}-\underset{\sim}{\mathbf{k}})\bullet\mathbf{x}}v_k(\eta)v_{\underset{\sim}{k}}^*(\eta)\hat{a}_{\mathbf{k}}^+\hat{a}_{\underset{\sim}{\mathbf{k}}}^- + e^{-i(\mathbf{k}+\underset{\sim}{\mathbf{k}})\bullet\mathbf{x}}v_{\underset{\sim}{k}}(\eta)v_{\underset{\sim}{k}}(\eta)\hat{a}_{\mathbf{k}}^+\hat{a}_{\underset{\sim}{\mathbf{k}}}^+ \end{pmatrix} \tag{68}$$

$$= \frac{1}{4}\int d^3\mathbf{k}\int d^3\underset{\sim}{\mathbf{k}}\, m_{\text{eff}}^2(\eta)\begin{pmatrix} \delta(\mathbf{k}+\underset{\sim}{\mathbf{k}})v_k^*(\eta)v_{\underset{\sim}{k}}^*(\eta)\hat{a}_{\mathbf{k}}^-\hat{a}_{\underset{\sim}{\mathbf{k}}}^- + \delta(\mathbf{k}-\underset{\sim}{\mathbf{k}})v_k^*(\eta)v_{\underset{\sim}{k}}(\eta)\hat{a}_{\mathbf{k}}^-\hat{a}_{\underset{\sim}{\mathbf{k}}}^+ \\ + \delta(\mathbf{k}-\underset{\sim}{\mathbf{k}})v_k(\eta)v_{\underset{\sim}{k}}^*(\eta)\hat{a}_{\mathbf{k}}^+\hat{a}_{\underset{\sim}{\mathbf{k}}}^- + \delta(\mathbf{k}+\underset{\sim}{\mathbf{k}})v_{\underset{\sim}{k}}(\eta)v_{\underset{\sim}{k}}(\eta)\hat{a}_{\mathbf{k}}^+\hat{a}_{\underset{\sim}{\mathbf{k}}}^+ \end{pmatrix}$$

$$= \frac{1}{4}\int d^3\mathbf{k}\, m_{\text{eff}}^2(\eta)\begin{pmatrix} +v_k^*(\eta)v_k^*(\eta)\hat{a}_{\mathbf{k}}^-\hat{a}_{-\mathbf{k}}^- + v_k^*(\eta)v_k(\eta)\hat{a}_{\mathbf{k}}^-\hat{a}_{\mathbf{k}}^+ \\ + v_k(\eta)v_k^*(\eta)\hat{a}_{\mathbf{k}}^+\hat{a}_{\mathbf{k}}^- + v_k(\eta)v_k(\eta)\hat{a}_{\mathbf{k}}^+\hat{a}_{-\mathbf{k}}^+ \end{pmatrix} \tag{69}$$

$$= \frac{1}{4}\int d^3\mathbf{k}\, m_{\text{eff}}^2(\eta)\left(\left(v_k^*(\eta)\right)^2\hat{a}_{\mathbf{k}}^-\hat{a}_{-\mathbf{k}}^- + v_k^*(\eta)v_k(\eta)\left(\hat{a}_{\mathbf{k}}^-\hat{a}_{\mathbf{k}}^+ + \hat{a}_{\mathbf{k}}^+\hat{a}_{\mathbf{k}}^-\right) + \left(v_k(\eta)\right)^2\hat{a}_{\mathbf{k}}^+\hat{a}_{-\mathbf{k}}^+\right).$$

Or, finally, using M&W (6.11) of pg. 66, to get the last line, we have

$$\begin{aligned} \hat{H}(\eta)_{3\text{rd term}} &= \frac{1}{4}\int d^3\mathbf{k}\, m_{\text{eff}}^2(\eta)\left(\left(v_k^*(\eta)\right)^2\hat{a}_{\mathbf{k}}^-\hat{a}_{-\mathbf{k}}^- + \left|v_k(\eta)\right|^2\left(2\hat{a}_{\mathbf{k}}^+\hat{a}_{\mathbf{k}}^- + \delta^{(3)}(0)\right) + \left(v_k(\eta)\right)^2\hat{a}_{\mathbf{k}}^+\hat{a}_{-\mathbf{k}}^+\right) \\ &= \frac{1}{4}\int d^3\mathbf{k}\left(\omega_k^2 - \mathbf{k}^2\right)\left(\left(v_k^*(\eta)\right)^2\hat{a}_{\mathbf{k}}^-\hat{a}_{-\mathbf{k}}^- + \left|v_k(\eta)\right|^2\left(2\hat{a}_{\mathbf{k}}^+\hat{a}_{\mathbf{k}}^- + \delta^{(3)}(0)\right) + \left(v_k(\eta)\right)^2\hat{a}_{\mathbf{k}}^+\hat{a}_{-\mathbf{k}}^+\right). \end{aligned} \quad .(70)$$

Adding all terms in (6.19), i.e., (62), (66), and (70), we have

$$\hat{H}(\eta) =$$

$$\frac{1}{4}\int d^3\mathbf{k}\begin{pmatrix} \left(v_k^{*\prime}(\eta)\right)^2\hat{a}_{\mathbf{k}}^-\hat{a}_{-\mathbf{k}}^- + \mathbf{k}^2\left(v_k^*(\eta)\right)^2\hat{a}_{\mathbf{k}}^-\hat{a}_{-\mathbf{k}}^- + \left(\omega_k^2 - \mathbf{k}^2\right)\left(v_k^*(\eta)\right)^2\hat{a}_{\mathbf{k}}^-\hat{a}_{-\mathbf{k}}^- \\ + \left|v_k'(\eta)\right|^2\left(2\hat{a}_{\mathbf{k}}^+\hat{a}_{\mathbf{k}}^- + \delta^{(3)}(0)\right) + \mathbf{k}^2\left|v_k(\eta)\right|^2\left(2\hat{a}_{\mathbf{k}}^+\hat{a}_{\mathbf{k}}^- + \delta^{(3)}(0)\right) \\ + \left(\omega_k^2 - \mathbf{k}^2\right)\left|v_k(\eta)\right|^2\left(2\hat{a}_{\mathbf{k}}^+\hat{a}_{\mathbf{k}}^- + \delta^{(3)}(0)\right) \\ + \left(v_k'(\eta)\right)^2\hat{a}_{\mathbf{k}}^+\hat{a}_{-\mathbf{k}}^+ + \mathbf{k}^2\left(v_k(\eta)\right)^2\hat{a}_{\mathbf{k}}^+\hat{a}_{-\mathbf{k}}^+ \\ + \left(\omega_k^2 - \mathbf{k}^2\right)\left|v_k(\eta)\right|^2\left(2\hat{a}_{\mathbf{k}}^+\hat{a}_{\mathbf{k}}^- + \delta^{(3)}(0)\right)\left(v_k(\eta)\right)^2\hat{a}_{\mathbf{k}}^+\hat{a}_{-\mathbf{k}}^+ \end{pmatrix} \tag{71}$$

$$= \frac{1}{4}\int d^3\mathbf{k}\begin{pmatrix} \left(v_k^{*\prime}(\eta)\right)^2\hat{a}_{\mathbf{k}}^-\hat{a}_{-\mathbf{k}}^- + \omega_k^2\left(v_k^*(\eta)\right)^2\hat{a}_{\mathbf{k}}^-\hat{a}_{-\mathbf{k}}^- \\ + \left|v_k'(\eta)\right|^2\left(2\hat{a}_{\mathbf{k}}^+\hat{a}_{\mathbf{k}}^- + \delta^{(3)}(0)\right) + \omega_k^2\left|v_k(\eta)\right|^2\left(2\hat{a}_{\mathbf{k}}^+\hat{a}_{\mathbf{k}}^- + \delta^{(3)}(0)\right) \\ + \left(v_k'(\eta)\right)^2\hat{a}_{\mathbf{k}}^+\hat{a}_{-\mathbf{k}}^+ + \omega_k^2\left(v_k(\eta)\right)^2\hat{a}_{\mathbf{k}}^+\hat{a}_{-\mathbf{k}}^+ \end{pmatrix}, \tag{72}$$

or finally,

$$\hat{H}(\eta)=\frac{1}{4}\int d^3\mathbf{k}\left[\begin{array}{l}\underbrace{\left(\left(v_k^{*\prime}(\eta)\right)^2+\omega_k^2\right)}_{F_k^*}\hat{a}_{\mathbf{k}}^-\hat{a}_{-\mathbf{k}}^-+\underbrace{\left(\left(v_k'(\eta)\right)^2+\omega_k^2\right)}_{F_k}\hat{a}_{\mathbf{k}}^+\hat{a}_{-\mathbf{k}}^+\\ \qquad\qquad\qquad+\underbrace{\left(\left|v_k'(\eta)\right|^2+\omega_k^2\left|v_k(\eta)\right|^2\right)}_{E_k}\left(2\hat{a}_{\mathbf{k}}^+\hat{a}_{\mathbf{k}}^-+\delta^{(3)}(0)\right)\end{array}\right] \quad (6.34)\text{ in M\&W} \qquad (73)$$